Lecture 6:

Recall: Many times we need to approximate $f(x)$ by:

$$f(x) = \sum_{k=0}^{N} a_k \cos kx + b_k \sin kx \quad \text{where}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \;; \quad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx \;; \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$$

Definition: (Real Fourier Series)

Consider $f(x) \in V = \{$ real-valued 2π-periodic smooth functions $\}$.

Then, the real Fourier Series of $f(x)$ is given by:

$f(x) = \sum_{k=0}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$, where $\{a_k\}$ and $\{b_k\}$ are given

by: $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$; $a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,dx$; $b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,dx$

Definition: (Complex Fourier Series)

Consider $f(x) \in W = \{\text{complex-valued } 2\pi\text{-periodic smooth functions}\}$

Then, the complex Fourier Series is given by:

$$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{ikx} \quad \text{where } \{C_k\} \text{ is determined by:}$$

$$C_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-\overset{\sqrt{-1}}{\overset{\shortparallel}{i}}kx}\, dx \quad \left(\text{Here, } e^{ikx} = \cos kx + i\sin kx\right)$$

The integration is computed separately for the real part and imaginary part.

Example: Consider: $u_{tt} = u_{xx}$ where $x \in (0, \pi)$, $t > 0$ such that

$$\begin{cases} u(0,t) = 0, & u(\pi, t) = 0 \\ u(x,0) = \phi(x), & u_t(x,0) = \psi(x) \end{cases}$$

Solution: Assume $u(x,t) = X(x)\,T(t)$.

Consider $L = \frac{\partial^2}{\partial x^2}$. Then, we choose $\{\phi_n(x)\}_{n=1}^{\infty} = \{\sin nx, \cancel{\cos nx}\}_{n=0}^{\infty}$.

$\because u(0,t) = u(\pi, t) = 0 \quad \therefore X(0) = X(\pi)$. We neglect $\cos nx$'s.

We can let $u(x,t) = \sum_{n=1}^{N} a_n(t) \sin nx$.

$$u_{tt} = u_{xx} \Rightarrow \sum_{n=1}^{N} a_n''(t) \sin nx = \sum_{n=1}^{N} (-n^2)\, a_n(t) \sin nx$$

Comparing coefficients $\Rightarrow a_n''(t) = -n^2 a_n(t)$.

$\because \quad a_n''(t) = -n^2 a_n(t)$.

$\therefore \quad a_n(t) =$ eigenfunction with eigenvalue $-n^2$.

Not:

$\sum (a_n \cos nt + b_n \sin nt)$

$$\therefore \quad \boxed{a_n(t) = a_n \cos nt + b_n \sin nt \qquad (a_n, b_n \in \mathbb{R})}$$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt) \sin nx$$

Now, a_n and b_n can be determined by initial condition:

$$u(x,0) = \phi(x) = \sum_{n=1}^{N} a_n \sin nx \Rightarrow a_n = \frac{2}{\pi}\int_0^{\pi} \phi(x) \sin nx\, dx$$

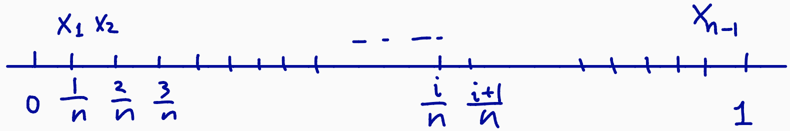
$$u_t(x,0) = \psi(x) = \sum_{n=1}^{N} n b_n \sin nx \Rightarrow b_n = \frac{2}{n\pi}\int_0^{\pi} \psi(x) \sin nx\, dx$$

(WHY?)

(Check!)

$$\frac{d^2 f}{dx^2}(x) = g(x) \qquad 0 < x < 1 \qquad \text{with} \quad f(0) = 1 \;;\; f(1) = 2 .$$

Discretize $(0, 1)$:



Approximation of $\frac{d^2 f}{dx^2}$:

$$f(x_{i+1}) \approx f(x_i) + \frac{1}{n} f'(x_i) + \frac{1}{2!}\frac{1}{n^2} f''(x_i)$$

(Taylor's expansion)

$$+ \quad f(x_{i-1}) \approx f(x_i) - \frac{1}{n} f'(x_i) + \frac{1}{2!}\frac{1}{n^2} f''(x_i)$$

$$f(x_{i+1}) + f(x_{i-1}) \approx 2f(x_i) + \frac{1}{n^2} f''(x_i)$$

$$\therefore \quad \frac{d^2 f}{dx^2}(x_i) \approx \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{\left(\frac{1}{n^2}\right)}$$

$$\therefore \quad \frac{d^2 f}{dx^2}(x_i) = g(x_i) \iff \left\{ \begin{array}{l} \vdots \\ \dfrac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{\left(\frac{1}{n^2}\right)} = g(x_i) \\ \vdots \end{array} \right.$$

$$D\vec{f} = \vec{g} \;;\; \vec{f} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{n-1}) \end{pmatrix} ; \; \vec{g} = \begin{pmatrix} g(x_1) \\ \vdots \\ g(x_{n-1}) \end{pmatrix}$$

for $i = 1, 2, \ldots, n-1$

In discrete case, a differential equation can be discretized as:

$$D\vec{u} = \vec{g}$$

where $\vec{u} = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \\ u(x_N) \end{pmatrix}$ = values of u at N points $\{x_1, x_2, \ldots, x_N\}$

$\vec{g} = \begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{pmatrix}$ = values of g at N points $\{x_1, x_2, \ldots, x_N\}$

D = $N \times N$ matrice approximating the differential operator.

<u>Question:</u> Can we "transform" $\vec{u}$ and $\vec{g}$ to turn the (BIG) linear system to SIMPLE algebraic equation?

<u>Answer:</u> YES! Discrete Fourier Transform!!